12.1

(1) $\int 2x^2+3x-5\,dx = \frac{2}{3}x^3+\frac{3}{2}x^2-5x+C$

(2) $\int (x-1)\sqrt{x}\,dx = \int x^{\frac{3}{2}}-x^{\frac{1}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}+C$

(3) $\int \frac{(1-x)^2}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}-2x^{\frac{1}{2}}+x^{\frac{3}{2}}\,dx = 2x^{\frac{1}{2}}-\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$

(4) $\int \frac{x+3}{x+1}\,dx = \int 1+\frac{2}{x+1}\,dx = x+\ln(x+1)^2+C$

12.3.

解: Define $F_n(x)=\int_1^x \ln^n x\,dx$, then $F_0(x)=(x-1)$, $\forall n\geq 1$

$$
\begin{aligned}
F_n(x) &= \int_1^x \ln^n t\,dt \\
&= \int_1^x d(t\ln^n t) - t\,d(\ln^n t) \\
&= x\ln^n x - \int_1^x t\cdot n\cdot \ln^{n-1}t\cdot \frac{1}{t}\,dt \\
&= x\ln^n x - nF_{n-1}(x) \\
&= x\ln^n x - n\left(x\ln^{n-1}x-(n-1)F_{n-2}(x)\right) \\
&\cdots \\
&= x\ln^n x - nx\ln^{n-1}x + n(n-1)x\ln^{n-2}x+\cdots+(-1)^{n-1}\cdot n!\,(x\ln x - F_0(x)) \\
&= \sum_{k=0}^{n}(-1)^k A_n^k \ln^{n-k}x - (-1)^n\cdot n!
\end{aligned}
$$

where $A_n^k=\frac{n!}{(n-k)!}$.

Hence the primitive function of $\ln^n x$ is $\sum_{k=0}^{n}(-1)^k A_n^k \ln^{n-k}x + C$.

12.5

Pf: Set $F(x) := \int_x^{2x} \frac{dt}{\sqrt{t^4+t^2+1}}$. Since $t^4+t^2+1 \geq 1$, $\forall t \in \mathbb{R}$, $f(t) := \frac{1}{\sqrt{t^4+t^2+1}} \in C^\infty(\mathbb{R})$.

Thus $F(x) \in C^\infty(\mathbb{R})$, and.

$$F'(x) = \frac{2}{\sqrt{(2x)^4+(2x)^2+1}} - \frac{1}{\sqrt{x^4+x^2+1}}, \quad \forall x \in \mathbb{R}.$$

$F'(x) \geq 0 \Leftrightarrow 4(x^4+x^2+1) \geq (2x)^4+(2x)^2+1 = 16x^4+4x^2+1 \Leftrightarrow x^4 \leq \frac{1}{4} \Leftrightarrow -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}$

Thus $F(x)$ is monotone increasing on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, decreasing on $(-\infty, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, +\infty)$

Now we study the properties of $F(x)$ near $\pm\infty$. Since $F(x) = F(-x)$, we only consider the case when $x \to +\infty$, $\forall x \geq 0$, we have

$$0 \leq F(x) = \int_x^{2x} \frac{dt}{\sqrt{t^4+t^2+1}}$$

$$\leq \int_x^{2x} \frac{dt}{\sqrt{t^4}} = \frac{1}{2x} \to 0 \text{ as } x \to +\infty.$$

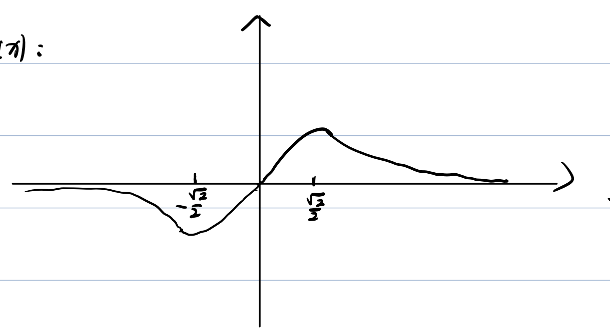
Therefore, we obtain the properties of $F(x)$ as below:

(i') $F \in C^\infty(\mathbb{R})$, $F(x) = -F(-x)$

(ii) $F'(x) > 0$ on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $F'(x) < 0$ on $(-\infty, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, +\infty)$

(iii) $\forall x \neq 0$, $F(x) > 0$. $F(0) = F(+\infty) = F(-\infty) = 0$

and we can draw the graph of $F(x)$:



12.7

Pf:

WLOG, we may assume that f is not identically 0 i.e. $\exists y_0 \in \mathbb{R}$ s.t. $f(y_0) \neq 0$.

$\forall x \in \mathbb{R}$, $f(x) = \frac{1}{f(y_0)} \int_{x-y_0}^{x+y_0} f(t)\,dt \in C^1(\mathbb{R})$ since $f(x) \in C(\mathbb{R})$ and

$$f'(x) = \frac{1}{f(y_0)}\left(f(x+y_0) - f(x-y_0)\right) \in C^1(\mathbb{R})$$

By induction, $f \in C^\infty(\mathbb{R})$. Now we differentiate $f(x)f(y) = \int_{x-y}^{x+y} f(t)\,dt$ w.r.t x and y respectively. $\forall (x,y) \in \mathbb{R}\times\mathbb{R}$, we have

$$f'(x)f(y) = f(x+y) - f(x-y) \quad (1)$$

and $$f(x)f'(y) = f(x+y) + f(x-y) \quad (2)$$

From (2), we have $f(y)f'(x) = f(x+y) + f(y-x)$. Combine it with (1), $f(x-y) = -f(y-x)$ i.e.

$$f(x) = f(-x), \ \forall x \in \mathbb{R}.$$

Differentiate (1), (2) again, we obtain

$$f''(x)f(y) = f'(x+y) - f'(x-y)$$

and $$f(x)f''(y) = f'(x+y) - f'(x-y)$$

Hence $f''(x)f(y) = f(x)f''(y)$, $\forall (x,y) \in \mathbb{R}\times\mathbb{R}$. Since $f(y_0) \neq 0$, we get

$$f''(x) = \frac{f''(y_0)}{f(y_0)} \cdot f(x).$$

Set $C = \frac{f''(y_0)}{f(y_0)}$, then $f''(x) = C \cdot f(x)$, $\forall x \in \mathbb{R}$. We will consider three cases:

① $C = 0$.

Now $f''(x) \equiv 0$, $f(x) = ax + b$ for some $a, b \in \mathbb{R}$, combine with $f(x) \equiv f(-x)$,

we have $f(x) = ax$, $\forall x \in \mathbb{R}$. Since

$$f(x)f(y) = a^2 xy, \quad \int_{x-y}^{x+y} f(t)\,dt = \frac{a}{2}t^2\Big|_{x-y}^{x+y} = 2axy,$$

we have $a = 2$, hence $f(x) \equiv 2x$.

② $C > 0$.

$\forall x \in \mathbb{R}$, $f''(x) = C f(x)$, then $f(x) = ae^{\sqrt{C}x} + be^{-\sqrt{C}x}$ for some $a, b, d \in \mathbb{R}$.

Since $f(x) \equiv -f(-x)$, we get $b = -a$ i.e. $f(x) = a(e^{\sqrt{C}x} - e^{-\sqrt{C}x}) \overset{a_0 = \frac{a}{2}}{=} a_0 \sinh(\sqrt{C}x)$

By $f(x)f(y) = a_0^2 \sinh(\sqrt{C}x)\sinh(\sqrt{C}y)$ and

$$\int_{x-y}^{x+y} f(t)\,dt = \frac{a_0}{\sqrt{c}}\cosh(\sqrt{c}t)\Big|_{x-y}^{x+y} = \frac{2a_0}{\sqrt{c}}\sinh(\sqrt{c}x)\sinh(\sqrt{c}y).$$

We have $a_0^2 = \frac{2a_0}{\sqrt{c}}$ i.e. $a_0 = \frac{2}{\sqrt{c}}$. Hence $f(x) = \frac{2}{\sqrt{c}}\sinh(\sqrt{c}x)$.

③ $c<0$.

Now $f(x) = a\sin(\sqrt{-c}x) + b\cos(\sqrt{-c}x)$, combine with $f(x) \equiv f(-x)$, we have $b=-a$ i.e. $f(x) = a\sin\sqrt{-c}x$.

By $f(x)f(y) = a^2\sin(\sqrt{-c}x)\sin(\sqrt{-c}y)$ and

$$\int_{x-y}^{x+y} f(t)\,dt = -\frac{a}{\sqrt{-c}}\cos(\sqrt{-c}t)\Big|_{x-y}^{x+y} = \frac{2a}{\sqrt{-c}}\sin(\sqrt{-c}x)\sin(\sqrt{-c}y).$$

We have $a = \frac{2}{\sqrt{-c}}$. Hence $f(x) = \frac{2}{\sqrt{-c}}\sin(\sqrt{-c}x)$.

(How to solve $\begin{cases} f''(x) = cf(x), & x \geq 0 \\ f(0) = 0 \end{cases}$? Here we consider the case $c>0$.

If $\exists\, x_0 > 0$ s.t. $f(x_0) \neq 0$. WLOG, we may assume that $x_0>0$ and $f(x_0)>0$.

Now we prove that $f(x) \geq 0$, $\forall x > 0$. Otherwise $\exists\, y_0 > 0$ s.t. $f(y_0) < 0$.

If $y_0 \in (0, x_0)$, then $\exists\, y_1 \in (0, x_0)$ s.t. $f(y_1) = \inf_{[0,x_0]} f < 0$, which contradicts $f''(y_1) \geq 0$

If $y_0 \in (x_0, +\infty)$, since $f(x_0) > 0$, $\exists\, y_1 \in (0, y_0)$ s.t. $f(y_1) = \sup_{[0,y_0]} f > 0$, which contradicts to $f''(y_1) \leq 0$.

Similarly, we can prove that $f(x)$ is monotone non-decreasing, then $f'(x) \geq 0$.

By $f''(x) = cf(x)$, we have $f''(x)f'(x) = cf'(x)f(x)$,

$$(f'(x))^2 - (f'(0))^2 = c(f(x))^2$$

If $f'(0) = 0$, we have $f'(x) = \sqrt{c}f(x) \Leftrightarrow (f(x)\cdot e^{-\sqrt{c}x})' = 0$

If $f'(0) \neq 0$, we have $\frac{f'(x)}{\sqrt{(f'(0))^2 + c(f(x))^2}} = 1 \overset{f'(0)>0}{\Leftrightarrow} \frac{d\left(\frac{\sqrt{c}f(x)}{f'(0)}\right)}{\sqrt{1+\left(\frac{\sqrt{c}f(x)}{f'(0)}\right)^2}} = \sqrt{c} \Leftrightarrow \ln\left(\frac{\sqrt{c}f(x)}{f'(0)} + \sqrt{1+\left(\frac{\sqrt{c}f(x)}{f'(0)}\right)^2}\right) = \sqrt{c}x$

$$\Leftrightarrow 1 + \left(\frac{\sqrt{c}f(x)}{f'(0)}\right)^2 = e^{2\sqrt{c}x} - 2e^{\sqrt{c}x}\cdot\frac{\sqrt{c}f(x)}{f'(0)} + \left(\frac{\sqrt{c}f(x)}{f'(0)}\right)^2$$

$$\Leftrightarrow f(x) = \frac{f'(0)}{2\sqrt{c}}\left(e^{\sqrt{c}x} - e^{-\sqrt{c}x}\right)$$

129. Additional assumption: $v \geq 0$

Pf:

$$u(x)e^{-\int_0^x v(t)\,dt} \le \left(C+\int_0^x u(t)v(t)\,dt\right)e^{-\int_0 v(t)\,dt}.$$

Define $F(x) := \left(C+\int_0^x u(t)v(t)\,dt\right)e^{-\int_0^x v(t)\,dt}$. Since

$$F'(x) = \left(u(x)v(x) - v(x)\left(C+\int_0^x u(t)v(t)\,dt\right)\right)e^{-\int_0^x v(t)\,dt} \le 0.$$

We have $F(x) \le F(0) = C$, $\forall x \in \mathbb{R}_+$. Hence

$$u(x)e^{-\int_0^x v(t)\,dt} \le F(x) \le F(0) = C.$$

12.11.

Pf:

Define $\alpha(t) = \begin{cases} \frac{\sin t}{t^2} - \frac{1}{t} = \frac{\sin t - t}{t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$

Since $\lim_{t\to 0} \alpha(t) = \lim_{t\to 0} \frac{\sin t - t}{t^2} = \lim_{t\to 0} \frac{\cos t - 1}{2t} = \lim_{t\to 0} \frac{-\sin t}{2} = 0$. Hence $\alpha(t) \in C(\mathbb{R})$.

We have $\lim_{x\to 0} \int_x^{3x} \alpha(t)\,dt = 0$.

$$\Rightarrow \quad \lim_{x\to 0} \int_x^{3x} \frac{\sin t}{t^2}\,dt = \lim_{x\to 0} \int_x^{3x} \frac{1}{t}\,dt$$

$$= \lim_{x\to 0} \ln 3 = \ln 3 \qquad \square.$$